An Improvement on the AM-GM Inequality/ A Further Exploration on a Cyclic Homogeneous Inequality

Harry Shen

March 17, 2023

1 Introduction

Chebyshev polynomials of the first kind are defined by the recurrence relation:

$$T_0(x) = 1,\ T_1(x) = x,\ T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \tag{1}$$

and have the explicit expression

$$T_n(x) = \frac{1}{2}\left(x - \sqrt{x^2-1}\right)^n + \frac{1}{2}\left(x + \sqrt{x^2-1}\right)^n, \tag{2}$$

for $x \geq 1$. The most well-known property of Chebyshev polynomials is that they express $\cos(n\theta)$ in terms of $\cos(\theta)$ via the equation $\cos(n\theta) = T_n(\cos\theta)$.

Chebyshev polynomials are a special case of Jacobi polynomials (also known as hypergeometric polynomials), a class of classical orthogonal polynomials. Chebyshev was the first mathematician to have noticed them in 1854, but their importance was not noticed until Hans Hahn rediscovered them and named them after Chebyshev. The polynomials $T_n(x)$ are orthogonal with respect to the inner product

$$\langle f, g\rangle = \frac{2}{\pi}\int_{-1}^{1} f(x)g(x)\frac{dx}{\sqrt{1-x^2}}.$$

In other words, $\langle T_m, T_n\rangle = 0$ for all positive integers $m \neq n$, and $\langle T_n, T_n\rangle = 1$ for all integers $n \geq 0$.

By replacing n with α in Equation (2), one can generalize Chebyshev polynomials to functions $T_\alpha : [-1, \infty) \to [0, \infty)$ for all values of $\alpha \in \mathbb{R}$. Equivalently, T_α is defined by

$$T_\alpha\left(\frac{x + x^{-1}}{2}\right) = \frac{x^\alpha + x^{-\alpha}}{2}, \tag{3}$$

for all $x > 0$.

From Equation (1), we can see by induction that $T_{n+1}(x) \geq T_n(x)$ for all $x \geq 1$. More generally, if $\alpha > \beta \geq 0$, then $T_\alpha(x) > T_\beta(x)$ for all $x > 1$. To see this, by virtue of Equation 3, we need to show that the function $\alpha \mapsto x^\alpha + x^{-\alpha}$ is a strictly increasing function of $\alpha > 0$ for a fixed positive $x \neq 1$. The claim then follows from

$$\frac{d}{d\alpha}(x^\alpha + x^{-\alpha}) = (x^\alpha - x^{-\alpha})\ln x > 0,$$

which holds for all positive $x \neq 1$ and $\alpha > 0$. Since $T_{-\alpha} = T_\alpha$, we have proven the following lemma.

Lemma 1. *If $|\alpha| \geq |\beta|$, then $T_\alpha(x) \geq T_\beta(x)$ for all $x \geq 1$. The equality occurs if and only if $x = 1$ or $|\alpha| = |\beta|$.*

Next, we prove an inequality that gives an upper bound for products of Chebyshev polynomials in terms of another Chebyshev polynomial.

Theorem 1. *If $2\alpha^2 \geq \beta^2 + \gamma^2$, then*

$$(T_\alpha(x))^2 \geq T_\beta(x)T_\gamma(x),$$

for all $x \geq 1$. The equality occurs if and only if $x = 1$ or $|\alpha| = |\beta| = |\gamma|$.

Proof. We let

$$G(x) = (x^{\alpha} + x^{-\alpha})^2 - (x^{\beta} + x^{-\beta})(x^{\gamma} + x^{-\gamma}),$$

and $H(x) = xG'(x)$. Then, for all $x > 0$, we have

$$\begin{aligned} xH'(x) &= 4\alpha^2(x^{2\alpha} + x^{-2\alpha}) - (\beta+\gamma)^2(x^{\beta+\gamma} + x^{-\beta-\gamma}) - (\beta-\gamma)^2(x^{\beta-\gamma} + x^{-\beta+\gamma}) \\ &\geq 4\alpha^2(x^{2\alpha} + x^{-2\alpha}) - (\beta+\gamma)^2(x^{2\alpha} + x^{-2\alpha}) - (\beta-\gamma)^2(x^{2\alpha} + x^{-2\alpha}) \\ &\geq 2(2\alpha^2 - \beta^2 - \gamma^2)(x^{2\alpha} + x^{-2\alpha}) \geq 0, \end{aligned}$$

since $|2\alpha| \geq |\beta + \gamma|$ and $|2\alpha| \geq |\beta - \gamma|$. It follows that $H'(x) \geq 0$ for all $x > 0$. Since $H(1) = 0$, we must have $H(x) \geq 0$ for all $x \geq 1$ and $H(x) \leq 0$ for all $0 < x \leq 1$. Therefore, $G'(x) \geq 0$ for all $x \geq 1$ and $G'(x) \leq 0$ for all $0 < x \leq 1$. Since $G(1) = 0$, it follows that $G(x) \geq 0$ for all $x > 0$. The equality occurs if and only if $x = 1$ or $|2\alpha| = |\beta + \gamma|$ or $|2\alpha| = |\beta - \gamma|$. Therefore, the equality occurs if and only if $x = 1$ or $|\alpha| = |\beta| = |\gamma|$. □

A function $f(x)$ is said to be *concave* on an interval $[a, b]$, if

$$f(tx_1 + (1-t)x_2) \geq tf(x_1) + (1-t)f(x_2),$$

for all $x_1, x_2 \in [a, b]$. A function $f(x)$ is said to be *midpoint-concave* on an interval $[a, b]$, if

$$f\left(\frac{x_1 + x_2}{2}\right) \geq \frac{f(x_1) + f(x_2)}{2},$$

for all $x_1, x_2 \in [a, b]$. It follows from Theorem 1 that the function $f_x : \mathbb{R} \to \mathbb{R}$ defined by

$$f_x(\alpha) = \ln T_{\sqrt{\alpha}}(x),$$

is a midpoint-concave function of α for a fixed value of $x \geq 1$. A theorem of Jensen states that if a function is continuous and midpoint-concave, then it is concave [5, 6]. It follows that f_x is a concave function. The next theorem is a generalization of Theorem 1.

Theorem 2. *Let t_i, $1 \le i \le n$, be nonnegative real numbers such that $\sum_{i=1}^n t_i = 1$. If $\alpha^2 \ge \sum_{i=1}^n t_i \alpha_i^2$, then we have*

$$T_\alpha(x) \ge \prod_{i=1}^n (T_{\alpha_i}(x))^{t_i}, \tag{4}$$

for all $x \ge 1$, where the equality occurs if and only if $x = 1$ or $|\alpha| = |\alpha_i|$ for all $1 \le i \le n$ with $t_i \ne 0$. Conversely, if the inequality (4) *holds for all $x \ge 1$, then $\alpha^2 \ge \sum_{i=1}^n t_i \alpha_i^2$.*

Proof. Since f_x is concave, by Jensen's inequality [4], we have

$$f_x\left(\sum_{i=1}^n t_i \alpha_i\right) \ge \sum_{i=1}^n t_i f_x(\alpha_i),$$

which implies the inequality (4). The converse is left to the reader as an exercise. □

2 A related cyclic homogeneous inequality

In the next theorem, we derive a cyclic homogeneous inequality on two variables based on Lemma 1.

Theorem 3. *Let $a, b \ge 0$ and $c = (a + b + 1)/2$. If $(c - 1)^2 \le 2ab$, then*

$$(x^c + y^c)^2 \ge (x + y)(x^a y^b + y^a x^b), \tag{5}$$

for all $x_1, x_2 \ge 0$, and the equality occurs if and only if $x = y$ or $\{a, b\} = \{0, 1\}$.

Proof. With $\alpha = c/2$, $\beta = 1/2$, and $\gamma = (a - b)/2$, Lemma 1 implies that

$$((x/y)^{c/2} + (x/y)^{-c/2})^2 \ge ((x/y)^{1/2} + (x/y)^{-1/2})((x/y)^{(a-b)/2} + (x/y)^{(b-a)/2}),$$

for all $x \geq 1$, if $2\alpha^2 \geq \beta^2 + \gamma^2$ or equivalently $(c-1)^2 \leq 2ab$. The equality occurs if and only if $x/y = 1$ or $|\alpha| = |\beta| = |\gamma|$, or equivalently, if and only if $x = y$ or $\{a, b\} = \{0, 1\}$. □

To generalize the inequality (5), we consider the following homogeneous cyclic inequality

$$\left(\sum_{i=1}^{n} x_i^c\right)^2 \geq \sum_{i=1}^{n} x_i \sum_{i=1}^{n} x_i^a x_{i+1}^b, \tag{6}$$

where $c = (a+b+1)/2$ and $a, b, x_1, \ldots, x_n \geq 0$. One asks that under what conditions on a, b, c, the inequality (6) holds for all $x_1, \ldots, x_n \geq 0$. It is straightforward to see that if $a + b = 1$, then the inequality (6)for all $x_1, \ldots, x_n \geq 0$, follows from the Rearrangement inequality [2, Ch. 6]. However, if $a + b \neq 1$, then the inequality (6) fails to hold if n is large enough [3]. In other words, the validity of the inequality (6) for all $x_1, \ldots, x_n \geq 0$ for fixed values of $a, b \geq 0$ with $a + b \neq 1$ depends on n. Conversely, given a fixed value of n, the inequality (6) holds for a specific subset of values $(a, b) \in [0, \infty) \times [0, \infty)$.

In the following theorem, we derive a sufficient condition for the inequality (6) in the case of $n = 3$.

Theorem 4. *If* $2a + 1 \geq b \geq (a-1)/2 \geq -b/2$*, then*

$$(x^c + y^c + z^c)^2 \geq (x + y + z)(x^a y^b + y^a z^b + z^a x^b),$$

for all $x, y, z \geq 0$*. The equality occurs if and only if* $x = y = z$*.*

Proof. Without loss of generality, we assume that $a \geq b$. If $b \geq a - 1$, then the claim follows from [3, Prop. 2.1]. Thus, suppose that $a - b - 1 \geq 0$. Let $x, y, z \geq 0$. By Jensen's inequality [2, Ch. 7]:

$$\frac{a+1-b}{2c} x^{2c} + \frac{b}{c} x^c y^c \geq x^{a+1} y^b,$$

$$\frac{b}{2c} x^{2c} + \frac{2b-a+1}{2c} y^{2c} + \frac{a-b}{c} x^c y^c \geq x^a y^{b+1},$$

$$\frac{a-b-1}{2c}x^{2c} + \frac{1}{c}x^c z^c + \frac{b}{c}x^c y^c \geq x^a y^b z.$$

Adding these inequalities yields

$$\frac{2a-b}{2c}x^{2c} + \frac{2b-a+1}{2c}y^{2c} + \frac{a+b}{c}x^c y^c + \frac{1}{c}x^c z^c \geq (x+y+z)x^a y^b. \tag{7}$$

Similarly

$$\frac{2a-b}{2c}y^{2c} + \frac{2b-a+1}{2c}z^{2c} + \frac{a+b}{c}y^c z^c + \frac{1}{c}x^c y^c \geq (x+y+z)y^a z^b, \tag{8}$$

$$\frac{2a-b}{2c}z^{2c} + \frac{2b-a+1}{2c}x^{2c} + \frac{a+b}{c}x^c z^c + \frac{1}{c}y^c z^c \geq (x+y+z)z^a x^b. \tag{9}$$

The claim follows from adding inequalities (7)-(9). □

3 The Case of $n = 8$

A particular case of interest is when $a = b = 1$ and $n = 8$. We first need a lemma.

Lemma 2. *Let $x_1, \ldots, x_8$ be nonnegative real numbers. Then*

$$\sum_{i=1}^{8} x_i^3 \geq \frac{1}{8}\left(\sum_{i=1}^{8} x_i\right)\left(\sum_{i=1}^{8} x_i^2\right).$$

Proof. By the Power Mean Inequality [1, Ch. III], one has

$$\left(\frac{1}{8}\sum_{i=1}^{8} x_i^3\right)^{1/3} \geq \frac{1}{8}\sum_{i=1}^{8} x_i \text{ and } \left(\frac{1}{8}\sum_{i=1}^{8} x_i^3\right)^{1/3} \geq \left(\frac{1}{8}\sum_{i=1}^{8} x_i^2\right)^{1/2}.$$

The claim follows from these inequalities. □

Theorem 5. *Let $x_1, \ldots, x_n$ be nonnegative real numbers. If $n \leq 8$, then*

$$\left(\sum_{i=1}^{n} x_i^3\right)^2 \geq \left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} x_i^2 x_{i+1}^2\right), \tag{10}$$

where the equality occurs if and only if $x_1 = x_2 = \cdots = x_n$. *Moreover, the inequality* (10) *does not hold in general if* $n > 8$.

Proof. Equivalently, we show that the maximum value of the function $f : \mathbb{U} \to \mathbb{R}$ defined by

$$f(x_1, \ldots, x_8) = \frac{\sum_{i=1}^{8} x_i^4 x_{i+1}^4}{\left(\sum_{i=1}^{8} x_i^6\right)^2},$$

is 1, where

$$\mathbb{U} = \left\{ (x_1, \ldots, x_8) : \sum_{i=1}^{8} x_i^4 = 1 \right\}.$$

By the Power Mean Inequality [1, Ch. III], one has $(\sum_{i=1}^{8} x_i^6/8)^{1/6} \geq (\sum_{i=1}^{8} x_i^4/8)^{1/4}$. Therefore, $\sum_{i=1}^{8} x_i^6 \geq \sqrt{1/8}$ and so the function f is bounded from above on $\mathbb{U}$, hence it attains a positive absolute maximum on the compact set $\mathbb{U}$, say at $(x_1, \ldots, x_8)$. Without loss of generality, we can assume $x_1, \ldots, x_8 \geq 0$. By the method of Lagrange multipliers, there exists a real number λ such that

$$\frac{1}{A^4}\left(4x_i^3(x_{i-1}^4 + x_{i+1}^4)A^2 - 2AB(6x_i^5)\right) = \lambda(4x_i^3), \ \forall i = 1, \ldots, 8, \tag{11}$$

where $A = x_1^6 + \cdots + x_8^6$ and $B = x_1^4 x_2^4 + \cdots + x_8^4 x_1^4$. Therefore,

$$x_i^4(x_{i-1}^4 + x_{i+1}^4)A - 3Bx_i^6 = \lambda A^3 x_i^4, \ \forall 1 \leq i \leq 8. \tag{12}$$

By summing the equations (22), we have $\lambda = -B/A^2$. We need to show that $A^2 \geq B$. On the contrary, suppose $B > A^2$, and we will derive a contradiction.

Equations (21) imply that, if $x_i \neq 0$, then

$$3Bx_i^2 = A(x_{i-1}^4 + x_{i+1}^4) + AB. \tag{13}$$

First, we show that $x_i \neq 0$ for all $i \in \{1, \ldots, 8\}$. On the contrary, and without loss of generality, suppose $x_8 = 0$ and $x_7 > 0$. Given $\epsilon \in (0, x_7)$, let

$$\delta = \delta(\epsilon) = (x_7^4 - (x_7 - \epsilon)^4)^{1/4},$$

such that $(x_7 - \epsilon)^4 + \delta^4 = x_7^4$, and so $\delta^3 \delta' = (x_7 - \epsilon)^3$. We define

$$F(\epsilon) = f(x_1, \ldots, x_6, x_7 - \epsilon, \delta) = \frac{B_\epsilon}{A_\epsilon^2},$$

and compute

$$F'(\epsilon) = \frac{4(x_7 - \epsilon)^3}{A_\epsilon^4} \left(A_\epsilon^2 (-x_6^4 - \delta^4 + (x_7 - \epsilon)^4 + x_1^4) + 3A_\epsilon B_\epsilon ((x_7 - \epsilon)^2 - \delta^2) \right).$$

It follows that

$$\begin{aligned} \lim_{\epsilon \to 0^+} F'(\epsilon) &= \frac{x_7^3}{A^4} (A^2(+x_7^4 + x_1^4) - A^2 x_6^2 + 3ABx_7^2) \\ &= \frac{x_7^3}{A^4} (A^2(x_1^4 + x_7^4) + A^2 B) > 0, \end{aligned} \tag{14}$$

where we have used equation (13) with $i = 7$ to obtain $-A^2 x_6^2 + 3ABx_7^2 = A^2 B$. The inequality (14) is a contradiction with the assumption that $F(\epsilon)$ attains a maximum as $\epsilon \to 0^+$. We conclude that $x_i > 0$ for all $i = 1, \ldots, 8$. In particular, equations (13) hold for all $i = 1, \ldots, 8$. In the rest of the proof, we let $y_i = x_i^2$. Hence, with $C = B/A$, the equations (13) turn into

$$3y_i C = y_{i-1}^2 + y_{i+1}^2 + B, \ \forall 1 \leq i \leq 8. \tag{15}$$

It follows that

$$\begin{aligned} 3(y_i + y_{i+4})C &= 2B + y_{i-1}^2 + y_{i+1}^2 + y_{i+3}^2 + y_{i+5}^2, \\ 3(y_{i+2} + y_{i+6})C &= 2B + y_{i+1}^2 + y_{i+3}^2 + y_{i+5}^2 + y_{i+7}^2, \end{aligned}$$

which imply that $y_j + y_{j+4} = y_{j+2} + y_{j+6}$ for all j, since $y_{i-1} = y_{i+7}$ as the indices are computed modulo 8. Therefore, there exist nonnegative real numbers r, s such that

$$y_1 + y_5 = y_3 + y_7 = r, \tag{16}$$

$$y_2 + y_6 = y_4 + y_8 = s. \tag{17}$$

Equations (15) imply that

$$\begin{aligned}
3(y_1 - y_3)C &= y_8^2 - y_4^2 \\
3(y_2 - y_4)C &= y_1^2 - y_5^2 \\
3(y_3 - y_5)C &= y_2^2 - y_6^2 \\
3(y_4 - y_6)C &= y_3^2 - y_7^2.
\end{aligned}$$

Let $\bar{y}_i = y_i - r/2$ if i is odd, and $\bar{y}_i = y_i - s/2$ if i is even. It follows that $\bar{y}_i + \bar{y}_{i+4} = 0$ for all i. Moreover, for i odd, we have

$$\begin{aligned}
3C(\bar{y}_i - \bar{y}_{i+4}) = 3C(y_i - y_{i+4}) &= y_{i-1}^2 - y_{i+1}^2 + y_{i+3}^2 - y_{i+5}^2 \\
&= (y_{i-1} - y_{i+3})(y_{i-1} + y_{i+3}) + (y_{i+1} - y_{i+5})(y_{i+1} + y_{i+5}) \\
&= 2\bar{y}_{i-1}s + 2\bar{y}_{i+1}s,
\end{aligned}$$

which implies that $\bar{y}_i = (\bar{y}_{i-1} + \bar{y}_{i+1})s/(3C)$ for odd i. Similarly, $\bar{y}_i = (\bar{y}_{i-1} + \bar{y}_{i+1})r/(3C)$ for even i. It then follows that

$$\bar{y}_i = \frac{s}{3C}(\bar{y}_{i-1} + \bar{y}_{i+1}) = \frac{rs}{9C^2}(\bar{y}_{i-2} + \bar{y}_i + \bar{y}_i + \bar{y}_{i+2}) = \frac{2rs}{9C^2}\bar{y}_i,$$

for odd i, and similarly for even i. We claim that $9C^2 \neq 2rs$. On the contrary, suppose

$9C^2 = 2rs$, and so, since $C = B/A > A$, we must have

$$6\sqrt{2}A < 6\sqrt{2}C \le 4\sqrt{rs} \le 2(r+s) \le \sum_{i=1}^{8} y_i. \tag{18}$$

However, by Lemma 2, we have $8A \ge \sum_{i=1}^{8} y_i$ which contradicts (18), since $6\sqrt{2} > 8$. Thus $9C^2 \ne 2rs$, and so $\bar{y}_i = 0$ for all i. Therefore, $y_1 = y_3 = y_5 = y_7 = r/2$, and $y_2 = y_4 = y_6 = y_8 = s/2$. So we have $r^2 + s^2 = 4((r/2)^2 + (s/2)^2) = \sum_{i=1}^{8} y_i^2 = 1$ and

$$f(x_1, \ldots, x_8) = \frac{8(r/2)^2(s/2)^2}{(4(r/2)^3 + 4(s/2)^3)^2} = \frac{2r^2 s^2}{(r^3+s^3)^2} \le 1, \tag{19}$$

since it follows from $r^2 + s^2 = 1$ that

$$r^3 + s^3 \ge 2\left(\frac{r^2+s^2}{2}\right)^{3/2} \ge 2\left(\frac{1}{2}\right)^{3/2} \ge \frac{r^2+s^2}{\sqrt{2}} \ge \sqrt{2}rs.$$

The equality occurs in (19) if and only if $r = s$ if and only if $y_1 = y_2 = \cdots = y_8$, hence the equality in (10) occurs if and only if $x_1 = x_2 = \cdots = x_8$. □

4 Variable Fractional Powers

It is difficult to determine when Equation (6) is satisfied for specific a,b,n. Exploring this inequality with variable fractional powers brings further bounds, bounds that include the AM-GM inequality.

Theorem 6. *Let $x_1, \ldots, x_n$ be nonnegative real numbers. If $\alpha \ge \frac{n}{2\sqrt{n-1}+n}$, then*

$$\sum_{i=1}^{n} x_i \ge n \prod_{x=1}^{n} x_i^{\frac{\alpha}{n} + \frac{(1-\alpha)x_i}{\sum_{i=1}^{n} x_i}}, \tag{20}$$

Proof. Equivalently, we show that the maximum value of the function $f : \mathbb{U} \to \mathbb{R}$

defined by

$$f(x_1,\ldots,x_n)=\sum_{x=1}^{n}(\frac{\alpha}{n}+(1-\alpha)x_i)\ln x_i,$$

is $\ln\frac{1}{n}$, where

$$\mathbb{U}=\left\{(x_1,\ldots,x_n):\sum_{i=1}^{n}x_i=1\right\}.$$

Without loss of generality, we can assume $x_1,\ldots,x_n\geq 0$. By the method of Lagrange multipliers, there exists a real number λ such that

$$\frac{\frac{\alpha}{n}+(1-\alpha)x_i}{x_i}+(1-\alpha)\ln x_i=\lambda\sum_{i=1}^{n}x_i,\ \forall i=1,\ldots,n, \tag{21}$$

where $\sum_{i=1}^{n}x_i$. Therefore,

$$\frac{\alpha}{nx_i}+(1-\alpha)+(1-\alpha)\ln x_i=\lambda,\ \forall 1\leq i\leq n. \tag{22}$$

There only exists a maximum or a minimum for λ when $x=\frac{\alpha}{(1-\alpha)n}$, so there exists at most 2 sets of value λ, which reduces Equation (6) to 2 variables. First we reexpress Equation (6) as 2 variables.

$$\frac{kx+(n-k)y}{n}\geq x^{k(\frac{\alpha}{n}+(1-\alpha)\frac{x}{kx+(n-k)y})}y^{k(\frac{\alpha}{n}+(1-\alpha)\frac{y}{kx+(n-k)y})} \tag{23}$$

Then homogenizing with $\frac{k}{n}=\beta$, we show

$$\ln(\beta x+1-\beta)\geq\left(\alpha\beta+\frac{\beta(1-\alpha)x}{\beta x+1-\beta}\right)\ln x,$$

for all $x\geq 1$.

Since there is an equality at $x=1$, we must compute that the derivative of the LHS is greater than the derivative of the RHS.

$$\frac{\beta}{\beta x + 1 - \beta} \geq \frac{\beta(1-\alpha)(1-\beta)}{(\beta x + 1 - \beta)^2} \ln x + \frac{\alpha\beta}{x} + \frac{(1-\alpha)\beta}{\beta x + 1 - \beta}$$
$$\frac{\alpha}{1-\alpha} \geq \frac{x \ln x}{(x-1)(\beta x + 1 - \beta)}$$

Consider $g_\beta(x) = \frac{x \ln x}{(x-1)(\beta x+1-\beta)}$. Notice that when β increases, g decreases when $\beta \geq \frac{1}{k}$. We must prove that the inequality holds at $\beta = \frac{1}{n}$, which would then show the inequality holds for all $k \geq 1$

$$\frac{nx \ln x}{(x-1)(x+n-1)} \leq \frac{\alpha}{1-\alpha} \tag{24}$$

The maximum of the LHS can be found by taking a derivative.

$$(1 + \ln x)(x^2 + (n-2)x - n + 1) - (2x^2 + (n-2)x) \ln x = 0$$
$$x^2 + (n-2)x - n + 1 - (x^2 + n - 1) \ln x = 0$$
$$\ln x = \frac{(x-1)(x+n-1)}{x^2 + n - 1}$$

We can substitute $\ln x$ into Equation (24):

$$\frac{n}{x + \frac{n-1}{x}} \leq \frac{\alpha}{1-\alpha} \tag{25}$$

In order to minimize the denominator to maximize the LHS, we set $x = \sqrt{n-1}$, which satisfies the bounds for α. $\square$

5 Conclusion

By [3, Prop. 2.2], if the inequality (10) holds in general for a value of n, then it holds in general for all smaller values of n. A counterexample for $n = 9$ was also provided in [3, §1]. Therefore, the inequality (10) holds for all $n \leq 8$.

It is generally difficult to determine if the inequality (6) holds for specific values of n, a, b. However, a necessary condition is that

$$(a + b - 1)^2 \leq 8ab \sin^2(\pi/n).$$

In particular, when $n = 2$, by Theorem 3, the inequality 6 holds with $n = 2$ in gneral if and only if $(a+b-1)^2 \leq 8ab$. For $n = 3$, in Theorem 4, we gave sufficient conditions on a, b so that the inequality 6 holds. It would be interesting to weaken these conditions or ideally determine exactly for what values of a, b the inequality (6) holds with $n = 3$.

Special cases of the inequality (6) can be verified using computer software. For example, it can be shown that the inequality (6) holds for $n = 3$, $a = 7/2$ and $b = 1/2$. In other words, one has

$$(x^5 + y^5 + z^5)^2 \geq (x^2 + y^2 + z^2)(x^7 y + y^7 z + z^7 x),$$

for all $x, y, , z \geq 0$. However, using a computer software seems unfeasible in general. The inequality (20) strengthens the AM-GM inequality, which can be shown when $n = 2$. Further explorations of this inequality can show other inequalities which can strengthen many of the base inequalities.

6 Acknowledgments

I would like to thank Dr. Javaheri at Siena University for mentoring me through this project.

References

[1] P. S. Bullen, *Handbook of Means and Their Inequalities*, Springer (2003).

[2] Z. Cvetkovski, Inequalities, *Theorems, Techniques, and Selected Problems*, Springer, Berlin (2012).

[3] M. Javaheri, A new arrangement inequality, *J. Ineq. Pure Appl. Math.* **7**(5) (2006), Article 162.

[4] G.H. Hardy, J.E. Littlewood, G. Pólya, Inequalities , Cambridge Univ. Press (1934).

[5] J. L. W. V. Jensen, Sur les fonctions convexes et les inégalités entre les valeurs moyennes, Acta Mathematica (Institut Mittag-Leffler) 1906, Vol. 30, No. 1, pp 175–193.

[6] C.P. Niculescu and L-E. Persson, Convex Functions and Their Applications, Springer International Publishing, (2018).