A Block Self-Dual Renormalization Group Procedure for the 1-D Quasi-Periodically Driven Ising Model

Solomon T. Zweig

December 2022

Abstract

In condensed matter theory, the Renormalization Group and the Quantum Ising Model have been used with great success to describe certain properties of condensed matter systems. The Quantum Ising Model is often used as a test-bed for many methods in condensed matter theory, including the Renormalization Group. In recent years two forms of the Renormalization group have been applied to the periodically driven Ising Model: The Strong Disorder Renormalization Group and the Block Self-Dual Renormalization Group. These efforts both showed that the Floquet system exhibited two new phases that are not present for the static system. However, the phase structure of a quasi-periodically driven (quasi-Floquet) Ising Model has not been studied with any Renormalization Group methods. Here it is shown that applying a Block Renormalization Group to a quasi-Floquet Ising Model in the high-frequency limit results in many possible sets of renormalized couplings and fields. Despite this, only a small number of them survive when the frequencies are taken to infinity, and the Renormalization Group Equations that result only differ by a multiplicative factor. These equations all ultimately lead to criticality conditions which match that for the static system up to an additive term and lead to the same correlation length critical exponent. This study is intended to serve as a starting point for using the Renormalization Group to examine quasi-Floquet Ising Models in a more general context and for using it to examine more complicated time-dependent systems.

1 Introduction

Understanding the behavior of many-particle quantum systems is one of the greatest challenges in modern physics. To face this challenge, many methods have been developed to determine the behavior of these systems. One of the most prominent is the Renormalization Group (RG). The essence of the RG is that it allows one to be able to examine a system at various length scales. One of the primary applications of the RG has been to determine the critical behavior of various systems, and their possible phases. One system that is of particular interest in this study is the one-dimensional (1-D) Quantum Ising Model. It is a rather simple many-body system, and as such it is often used as a test bed for new techniques in condensed matter theory. Among these techniques is the RG. Using RG methods, the phases the Ising Model possesses as well as the critical exponents of its phase transition can be determined [12, 7]. Such RG methods have also been used to deduce the critical behavior of other systems [14, 11, 8].

In recent years, there has been an increasing interest in driven systems, in particular periodically-driven (Floquet) systems [13, 5, 2, 6]. As such, new theoretical methods have been developed probe these systems, and methods used for static systems including the RG have been modified [1, 10]. One of the systems that has been used to test these Floquet methods is the Floquet Ising Model [6, 2, 1, 10]. And yet again, the RG has proven to be valuable tool.

Despite the success of the RG in the static and Floquet regimes [1, 10, 8, 9, 11, 14], little attention has been paid to the possibility of applying the RG to quasi-periodically driven (quasi-Floquet) systems. Therefore, in this study I develop an analytical RG procedure for quasi-periodically driven systems and test it on a 1-D quasi-Floquet Ising Model.

The remainder of this paper is organized as follows. In the next section, I provide some background information on critical exponents, the static 1-D Ising Model, Floquet theory and two versions of the RG. After that, I give an example of how the RG method used in this paper works on a simpler system. I then transition to applying that RG to a quasi-Floquet 1-D Ising Model.

2 Background

In this section, I give some background information on critical exponents, the 1-D Ising Model, Floquet theory, and two RG methods that are relevant to this study.

2.1 Critical exponents

In the theory of critical phenomena [12], some of the most important entities are critical exponents. They describe how physical quantities diverge near the critical point. These quantities generally diverge as power-law singularities of the difference of the control parameters (temperature, for example) and their critical values; I will denote this critical value by T_c. For example, in magnetic materials [12]

$$\chi \propto |t|^{-\gamma} \quad (T > T_c), \quad |t|^{-\gamma'} \quad (T < T_c) \tag{1}$$

$$m \propto |t|^{\beta} \quad (T < T_c) \tag{2}$$

$$\xi \propto |t|^{-\nu} \quad (T > T_c), \quad |t|^{-\nu'} \quad (T < T_c), \tag{3}$$

where m is the magnetization, χ is the magnetic susceptibility, ξ is the correlation length, and $t = (T - T_c)/T_c$. The magnetization conveys information about how many spins are aligned (the strength of a magnet), and the magnetic susceptibility determines how the magnetization responds to the presence of an external field magnetic field. The correlation length describes the size of fluctuations within a system. At the critical point, fluctuations of all sizes exist and the correlation length diverges. Of particular interest in this study is ν, which describes this divergence.

Another reason why critical exponents are important is because they define what are known as universality classes [12]. A universality class is a group of systems that have have the same critical exponents: if you know the critical exponents for one system in a class then you know the critical exponents for all of the other systems in that class. For example, the 1-D Floquet and static Ising Models are in the same universality class [1, 10]. This particular example will become relevant later on, when I examine the Floquet Ising Model and the quasi-periodically driven (quasi-Floquet) Ising Model with the RG.

2.2 The static Ising Model, some Floquet theory, and the Renormalization Group

The 1-D Quantum Ising Model (QIM) has become a ubiquitous test bed for many techniques in condensed matter theory. The static 1-D QIM consists of a 1-D lattice of spin-$\frac{1}{2}$ particles with a constant lattice spacing. The Hamiltonian of the system is

$$H = -\sum_{n=0}^{N-1} J_n \sigma_n^z \sigma_{n+1}^z - \sum_{n=0}^{N} h_n \sigma_n^x. \tag{4}$$

Here $\sigma^{x,y,z}$ are the Pauli matrices, J_n and h_n are positive random variables.

For Floquet 1-D QIM the Hamiltonian is no longer time-independent. In order to determine the critical behavior of the Floquet QIM, the primary method that has been used is the RG in conjunction with propagator Floquet theory [10, 1]. In propagator Floquet theory, the evolution operator of the system over one period is written as

$$U(T) = \mathcal{T} \exp\left[-i \int_0^T H(t')\, dt'\right] = e^{-iH_F T}, \tag{5}$$

where T is the period, $\mathcal{T}$ denotes time-ordering, and H_F is called the Floquet Hamiltonian. In most situations it is very challenging to find H_F, however for the two Floquet studies mentioned [1, 10] this task was actually very simple. Once the Floquet Hamiltonian has been found an RG procedure can be implemented.

Two analytical RG procedures have been used previously: the Block Self-Dual RG and the Strong-Disorder RG [1, 10]. In the Block Self-Dual RG all of the even or odd couplings and fields are renormalized at once, while in the Strong-Disorder RG only the strongest coupling or field is renormalized. As a result of such RG procedures, RG equations are obtained. These describe how the couplings/fields change as a result of the RG transformations. Using these two RG methods as well as another method, it has been shown [1, 10, 5] that there are two phases present for the Floquet system that are not present for the static system.

Although there have been a many studies done on Floquet systems [1, 2, 5, 6, 10, 13], not as much attention has been paid to quasi-Floquet systems. There have been a few studies on the possible phases of quasi-Floquet systems [4, 3], but there has not been any RG studies on such systems. In this study the Block Self-Dual RG is adapted for the quasi-periodically driven (quasi-Floquet) 1-D QIM. (From now on I will simply call this the RG). I use this RG to examine the possible phases it can be in from a new perspective and to see whether the results obtained are with prior results.

In order to analytically implement the RG for the quasi-Floquet system, the generalized Floquet-Magnus expansion (GFME) [4, 3] along with three limits are needed: the short-time, high-frequency, and weak-driving limits. The short-time limit is needed to be able to break up the evolution operator, as will be shown in 4.1. Because I work in the short-time limit, I have to take the high-frequency limit. If I did not do this, the driving would not have much of an effect on the system. Also the weak-driving limit serves a similar function as the short-time limit. Without these limits, implementing the RG analytically would be very challenging.

In the following section, I demonstrate how the RG works on a simpler system: the Floquet 1-D Ising Model. For this system, neither the GFME nor the three limits mentioned above are needed.

3 A Floquet example

As a simple example of how the RG procedure works, I detail the RG methods implemented in [10]. The system examined in that study was

$$H(t) = \begin{cases} -\sum_{n=1}^{N-1} J_n \sigma_n^z \sigma_{n+1}^z & 0 \le t \le T_0 \\ -\sum_{n=1}^{N} h_n \sigma_n^x & T_0 \le t \le T_0 + T_1 = T, \end{cases} \tag{6}$$

where T is the total period, the lattice spacing $b = 2$, and J_n and h_n are site dependent. To determine the lattice spacing b, denote the number of degrees of freedom before the RG is applied by N, and denote the number of degrees of freedom after the RG is applied by $N' = N/b^d$, where d is the dimension of the system. Because the number of degrees of freedom is reduced by half, when this RG is applied to the system, and because the dimension of the system is $d = 1$, $b = 2$. The reason why this is mentioned here will be explained at the end of the section.

In order to determine how the Floquet dynamics effect the phases, the RG is applied to the evolution operator over one period. This operator takes the following form:

$$U(T,0) = U(T,T_0)U(T_0,0) = \exp\left[iT_1 \sum_{n=1}^{N} h_n \sigma_n^x\right] \exp\left[iT_0 \sum_{n=1}^{N-1} J_n \sigma_n^z \sigma_{n+1}^z\right]. \tag{7}$$

Clearly, working with a Hamiltonian such as (6) makes the calculation of the evolution operator far more simple, that it would be with a continuous Hamiltonian.

In order to make progress, this operator can be factored into a product of terms with even and odd couplings:

$$U(T,0) = \exp\left[iT_1 \sum_{n=1}^{N/2} h_{2n}\sigma_{2n}^x\right] \exp\left[iT_1 \sum_{n=1}^{N/2} h_{2n-1}\sigma_{2n-1}^x\right] \times \exp\left[iT_0 \sum_{n=1}^{(N-1)/2} J_{2n-1}\sigma_{2n-1}^z\sigma_{2n}^z\right] \exp\left[iT_0 \sum_{n=1}^{(N-1)/2} J_{2n}\sigma_{2n}^z\sigma_{2n+1}^z\right]. \tag{8}$$

Using this form two terms can be introduced: $\mathcal{M}$ and $\mathcal{N}$. These are called the intra and inter-block terms respectively, and are given by

$$\mathcal{M} = \exp\left[iT_1 \sum_{n=1}^{N/2} h_{2n-1}\sigma_{2n-1}^x\right] \exp\left[iT_0 \sum_{n=1}^{(N-1)/2} J_{2n-1}\sigma_{2n-1}^z\sigma_{2n}^z\right] \tag{9}$$

and

$$\mathcal{N} = \exp\left[iT_0 \sum_{n=1}^{(N-1)/2} J_{2n}\sigma_{2n}^z\sigma_{2n+1}^z\right] \exp\left[iT_1 \sum_{n=1}^{N/2} h_{2n}\sigma_{2n}^x\right]. \tag{10}$$

$\mathcal{M}$ describes the couplings between spins in the same block, and $\mathcal{N}$ describes couplings between spins in adjacent blocks.

The next step is to determine the elements of $\mathcal{M}$:

$$\begin{aligned}\langle S_1,\cdots,S_N|\mathcal{M}|S_1',\cdots,S_N'\rangle &= \prod_{n=1}^{N/2} \delta_{S_{2n},S_{2n}'} \langle S_{2n-1}|e^{iT_1 h_{2n-1}\sigma_{2n-1}^x} e^{iT_0 J_{2n-1}\sigma_{2n-1}^z S_{2n}}|S_{2n-1}'\rangle \\ &= \prod_{n=1}^{N/2} \delta_{S_{2n},S_{2n}'} \langle S_{2n-1}|\mathcal{M}_{2n-1}(S_{2n})|S_{2n-1}'\rangle,\end{aligned} \tag{11}$$

where S_n and S_n' are spin values. Now, the eigenvalues and (more importantly) eigenvectors of $\mathcal{M}_{2n-1}(S_{2n})$ must be determined. Once the eigenvectors have been determined, a new set of basis vectors can be constructed out of tensor products of the even-spin eigenvectors and the eigenvectors of $\mathcal{M}_{2n-1}(S_{2n})$.

The eigenvectors of $\mathcal{M}_{2n-1}(S_{2n})$ are

$$|\lambda_{2n-1}^+(S_{2n})\rangle = e^{-\frac{iT_0 J_{2n-1}S_{2n}}{2}} \sqrt{\frac{1+r_{2n-1}S_{2n}}{2}}|S_{2n-1}=+\rangle + \eta_{2n-1} e^{\frac{iT_0 J_{2n-1}S_{2n}}{2}} \sqrt{\frac{1-r_{2n-1}S_{2n}}{2}}|S_{2n-1}=-\rangle \tag{12}$$

and

$$|\lambda_{2n-1}^-(S_{2n})\rangle = -\eta_{2n-1} e^{-\frac{iT_0 J_{2n-1}S_{2n}}{2}} \sqrt{\frac{1-r_{2n-1}S_{2n}}{2}}|S_{2n-1}=+\rangle + e^{\frac{iT_0 J_{2n-1}S_{2n}}{2}} \sqrt{\frac{1+r_{2n-1}S_{2n}}{2}}|S_{2n-1}=-\rangle, \tag{13}$$

where

$$r_{2n-1} = \frac{\sin T_0 J_{2n-1}}{\sqrt{\sin^2 T_0 J_{2n-1} + \tan^2 T_1 h_{2n-1}}} \tag{14}$$

and

$$\eta_{2n-1} = \mathrm{sign}\left(\frac{\tan T_1 h_{2n-1}}{\sin T_0 J_{2n-1}}\right). \tag{15}$$

Using these eigenvectors and the even-spin eigenvectors a new set of basis vectors can be defined:

$$|M_{S_2,\cdots,S_{2n}}^{(s_1,\cdots,s_{2n-1})}\rangle = \bigotimes_{n=1}^{N/2} |\lambda_{2n-1}^{s_{2n-1}}(S_{2n})\rangle \otimes |S_{2n}\rangle, \tag{16}$$

where s_{2n-1} is a pseudo-spin that labels the eigenvectors of each $\mathcal{M}_n$.

Now that this new basis has been defined, the next step is to determine the matrix elements of $\mathcal{N}$, in the new basis. The matrix elements are

$$\langle M^{(s_1,\cdots,s_{2n-1})}_{S_2,\cdots,S_{2n}}|\mathcal{N}|M^{(s_1,\cdots,s_{2n-1})}_{S_2,\cdots,S_{2n}}\rangle = \prod_{n=1}^{N/2}\cos(T_0J_{2n-2})\cos(T_1h_{2n})\big[\delta_{S_{2n},S'_{2n}}(1+ir_{2n-1}s_{2n-1}S_{2n-2}S_{2n}\tan T_0J_{2n-2}) + i\delta_{S_{2n},-S'_{2n}}\sqrt{1-r^2_{2n-1}}\tan(T_1h_{2n})\cos(T_0J_{2n-1})(1-S_{2n-2}S_{2n}\tan(T_0J_{2n-2})\tan(T_0J_{2n-1}))\big]. \quad (17)$$

(If the reader wants to go through the details of how the equation above was obtained, refer to [10].) This term is then matched with a similar term:

$$\langle S_2,\cdots,S_{2n}|\mathcal{N}^R|S'_2,\cdots,S'_{2n}\rangle = \prod_n \cos(T_0J^R_{2n-2,2n})\cos(T_1h^R_{2n})[\delta_{S_{2n},S'_{2n}}(1+i\tan(T_0J^R_{2n-2})S_{2n-2}S_{2n} + i\delta_{S_{2n},-S'_{2n}}\tan(T_1h^R_{2n})(1+i\tan(T_0J^R_{2n-2})S_{2n-2}S_{2n}], \quad (18)$$

which contains the renormalized couplings and fields for the even sites. (From now on, these will be called the renormalized parameters.)

The resulting renormalized parameters J^R and h^R are defined by

$$\tan(T_0J^R_{2n-2}) = s_{2n-1}\tan(T_0J_{2n-2})r_{2n-1} \quad \text{and} \quad \tan(T_1h^R_{2n}) = \tan(T_1h_{2n})\cos(T_0J_{2n-1})\sqrt{1-r^2_{2n-1}}, \quad (19)$$

and the RG equation,

$$\frac{|\tan(T_0J^R_{2n-2})|}{|\tan(T_1h^R_{2n})|} = \frac{|\tan(T_0J_{2n-2})\tan(T_0J_{2n-1})|}{|\tan(T_1h_{2n})\tan(T_1h_{2n-1})|}, \quad (20)$$

follows. Using

$$k_n = \frac{|\tan(T_0J_{n-1})|}{|\tan(T_1h_n)|} \quad \text{and} \quad k^R_n = \frac{|\tan(T_0J^R_{n-2,n})|}{|\tan(T_1h^R_n)|} \quad (21)$$

we make the following definitions:

$$k^R_{2n} = \frac{|\tan(T_0J^R_{2n-2,2n})|}{|\tan(T_1h^R_{2n})|} \quad \text{and} \quad k_{2n-1}k_{2n} = \frac{|\tan(T_0J_{2n-2})\tan(T_0J_{2n-1})|}{|\tan(T_1h_{2n-1})\tan(T_1h_{2n})|}, \quad (22)$$

the RG equation can be written as

$$k^R_{2n} = k_{2n-1}k_{2n}, \quad (23)$$

or equivalently as

$$\ln k^R_{2n} = \ln k_{2n-1} + \ln k_{2n}. \quad (24)$$

This condition is of the same form as that for the static system (see [9] and references therein). For that system $k_n = h_n/J_n$, and the average

$$\mu' = \frac{1}{N}\sum_{i=1}^{N}\ln\frac{h_i}{J_i} \quad (25)$$

characterizes the phases and is used to obtain the typical correlation length critical exponent as shown in [7]. (For the remainder of this paper, this will just be called the critical exponent.) There, the system is in the paramagnetic phase if $\mu' > 0$ and is in the ferromagnetic phase if $\mu' < 0$. The critical point is that for which $\mu' = \mu_c = 0$. The typical correlation length exponent is determined by noting that the typical correlation length can be written as

$$\xi_{typ} = |\mu - \mu_c|^{-\nu_{typ}} \quad (26)$$

near the critical point, and that the renormalized parameters are

$$J_n^R = \frac{J_{2n}J_{2n+1}}{\sqrt{(J_{2n+1})^2 + (h_{2n+1})^2}} \quad \text{and} \quad h_n^R = \frac{h_{2n-1}h_{2n}}{\sqrt{(J_{2n-1})^2 + (h_{2n-1})^2}}. \tag{27}$$

With these,

$$\mu'^R = \frac{2}{N}\sum_{i=1}^{N/2} \ln \frac{h_i^R}{J_i^R} = 2\mu, \tag{28}$$

and since the renormalized typical correlation length is

$$\xi_{typ}^R = \frac{\xi_{typ}}{b^d} = \frac{\xi_{typ}}{2}, \tag{29}$$

the critical exponent is $\nu_{typ} = 1$.

For the Floquet system, the possible phases are characterized by average

$$\mu = \frac{1}{N}\sum_{i=1}^{N} \ln \frac{|\tan(T_0 J_i)|}{|\tan(T_1 h_i)|} \tag{30}$$

For the paramagnetic phases, $\mu < 0$ and k_n flows to zero. For the ferromagnetic phases, $\mu > 0$ and k_n flows to infinity. The critical point is that for which $\mu = 0$. Therefore by using the same methods as shown above and by noting (29), it can be shown that $\nu_{typ}^F = 1$, where the F indicates that this exponent is for the Floquet system.

When the system is deep in any of these phases, k_n approaches zero or infinity. Thus, the possible phases are characterized by the possible of J_n and h_n that allow for this. Within these phases, all of the couplings, or fields are the same value. The only values of J and h that satisfy this are 0 and $\pi/2T_0$ and 0 and $\pi/2T_1$ respectively. These four values define the four possible phases. The phases for which $J = 0$ and $J = \pi/2T_0$ are known as the paramagnetic phase and the π-paramagnetic phase respectively. The phases for which $h = 0$ and $h = \pi/2T_1$ are known as the ferromagnetic phase and the π-ferromagnetic phase respectively. The π-paramagnetic phase and the π-ferromagnetic phase are unique to the Floquet system.

In the following sections, the RG procedure detailed above is adapted to a 1-D quasi-Floquet Ising Model in the short-time, high-frequency, weak-driving limit. In order to do this without having to deal with the additional complication of time-ordering, the GFME is used. Even with these approximations, the implementation of the RG is significantly complicated by the fact that the Hamiltonian varies continuously in time.

4 The GFME and breaking up the evolution operator

In this section, the implementation of the GFME for the quasi-Floquet Ising Model is presented. Such an expansion is appropriate for quasi-Floquet systems in the high-frequency limit. The reason why such a limit is taken here is the following. Because of the form of the evolution operator, it cannot be split into an intra-block term and an inter-block term without taking the short-time limit. Therefore in order for the driving to have a non-negligible effect, I take the high-frequency limit. This model also has a lattice spacing of $b = 2$ for the same reason mentioned in section 3.

The Hamiltonian of the model that is examined in this study is

$$H(t) = H_0 + V(t) = \sum_{n=1}^{N-1}(J_n + \mathcal{A}g(\tau_1))\sigma_n^z\sigma_{n+1}^z + \sum_{n=1}^{N}(h_n + \mathcal{B}f(\tau_2))\sigma_n^x, \tag{31}$$

where $\tau_{1,2} = \omega_{1,2}t$, ω_1 and ω_2 are incommensurate frequencies, and the driving amplitudes, $\mathcal{A}$ and $\mathcal{B}$ are weak. The reason why this Hamiltonian involves two frequencies is that the real-space RG only works if the eigenstates are localized in real-space, which for quasi-Floquet systems only happens with two frequencies.

With a Hamiltonian of this form it makes sense to go into the interaction picture. The reason for doing this is that by going into the interaction picture the evolution operator can be expressed as $U(t) = U_0(t)U_{int}(t)$ where

$$U_0(t) = \exp\left[-it\sum_n (J_n\sigma_n^z\sigma_{n+1}^z + h_n\sigma_n^x)\right] \quad \text{and} \quad U_{int}(t) = \mathcal{T}\exp\left[-i\int_0^t U_0(t')V(t')U_0^\dagger(t')\,dt'\right], \tag{32}$$

where $\mathcal{T}$ denotes time-ordering. By doing this, I have separated the time-independent and time-dependent parts of the Hamiltonian into two evolution operators. This is not necessary for the Floquet system, because of the form of the Hamiltonian.

Now, I only have to apply the GFME to $U_{int}(t)$. Due to the fact that the short time limit is being used, the integrand in $U_{int}(t)$ can be well approximated by $V - it[H_0, V] = V_{int}$. Because I am working in the high-frequency limit and because this is a quasi-Floquet system, the GFME can be applied to $U_{int}(t)$:

$$U_{int}(t) = P(t)e^{-iDt}P^\dagger(0), \tag{33}$$

where

$$P(t) = e^{-i\Gamma(t)} = \exp\left[-i\sum_{n=1}^\infty \Gamma^{(n)}(t)\right] \quad \text{and} \quad D = \sum_{n=1}^\infty D^{(n)}. \tag{34}$$

In the high-frequency limit, $P(t)$ generates the fast micromotion of system while e^{-iDt} generates the slow dynamics: D is an effective Hamiltonian.

For this study, I only consider $\Gamma^{(1)}$ and $D^{(1)} + D^{(2)} = D$, where

$$\Gamma^{(1)} = -i\sum_{\boldsymbol{n}\in\mathbb{Z}_2\neq 0} \frac{e^{\frac{i}{2}\boldsymbol{n}\cdot\boldsymbol{\omega}t}}{\boldsymbol{n}\cdot\boldsymbol{\omega}} V_{int,\boldsymbol{n}} \quad \text{and} \quad D = V_{int,0} + \frac{1}{2}\sum_{\boldsymbol{n}\in\mathbb{Z}_2\neq 0} \frac{[V_{int,\boldsymbol{n}}, V_{int,\boldsymbol{n}}^\dagger]}{\boldsymbol{n}\cdot\boldsymbol{\omega}}, \tag{35}$$

where $D^{(1)} = V_{int,0}$. Here, $\boldsymbol{n} = (n_1, n_2)$ and $\boldsymbol{\omega} = (\omega_1, \omega_2)$ are two-dimensional number and frequency vectors respectively, and $V_{int,\boldsymbol{n}}$ is the nth Fourier coefficient. Next, I use that

$$V_{int}(t) \approx \sum_n \left[\mathcal{A}g(\tau_1)\sigma_n^z\sigma_{n+1}^z + \mathcal{B}f(\tau_2)\sigma_n^x + 2t(J_n\mathcal{A}g(\tau_1) - h_n\mathcal{B}f(\tau_2))\sigma_n^y\sigma_{n+1}^z\right] \tag{36}$$

to give an explicit form of D. D can be written as $D = D^{(1)} + D^{(2)} = \sum_n D_n$, where

$$\begin{aligned} D_n = &\left[\mathcal{A}g_0 + 2iJ_n\mathcal{A}\mathcal{B}\sum_{\boldsymbol{n'}} \frac{1}{\boldsymbol{n'}\cdot\boldsymbol{\omega}}(f_{\boldsymbol{n'}}\tilde{g}_{\boldsymbol{n'}}^* - f_{\boldsymbol{n'}}^*\tilde{g}_{\boldsymbol{n'}})\right]\sigma_n^z\sigma_{n+1}^z \\ &+ \left[\mathcal{B}f_0 + 2ih_n\mathcal{A}\mathcal{B}\sum_{\boldsymbol{n'}} \frac{1}{\boldsymbol{n'}\cdot\boldsymbol{\omega}}(g_{\boldsymbol{n'}}\tilde{f}_{\boldsymbol{n'}}^* - g_{\boldsymbol{n'}}^*\tilde{f}_{\boldsymbol{n'}})\right]\sigma_n^x \\ &+ \left[2(J_n\mathcal{A}\tilde{g}_0 - h_n\mathcal{B}\tilde{f}_0) + i\mathcal{A}\mathcal{B}\sum_{\boldsymbol{n'}} \frac{1}{\boldsymbol{n'}\cdot\boldsymbol{\omega}}(g_{\boldsymbol{n'}}f_{\boldsymbol{n'}}^* - g_{\boldsymbol{n'}}^*f_{\boldsymbol{n'}})\right]\sigma_n^y\sigma_{n+1}^z \end{aligned} \tag{37}$$

and

$$\Gamma^{(1)} = -i\sum_{\boldsymbol{n'}\in\mathbb{Z}_2\neq 0,n} \frac{e^{\frac{i}{2}\boldsymbol{n'}\cdot\boldsymbol{\omega}t}}{\boldsymbol{n'}\cdot\boldsymbol{\omega}}\left(\mathcal{A}g_{\boldsymbol{n'}}\sigma_n^z\sigma_{n+1}^z + \mathcal{B}f_{\boldsymbol{n'}}\sigma_n^x + 2(J_n\mathcal{A}\tilde{g}_{\boldsymbol{n'}} - h_n\mathcal{B}\tilde{f}_{\boldsymbol{n'}})\sigma_n^y\sigma_{n+1}^z\right), \tag{38}$$

where $f_{\boldsymbol{n'}}$ and $g_{\boldsymbol{n'}}$ are the Fourier coefficients for $f(\tau_2)$ and $g(\tau_1)$ respectively, and $\tilde{f}_{\boldsymbol{n'}}$ and $\tilde{g}_{\boldsymbol{n'}}$ are the Fourier coefficients for $tf(\tau_2)$ and $tg(\tau_1)$ respectively.

The full evolution operator can now be written as

$$U(t) = U_0(t)e^{-i\Gamma^{(1)}(t)}e^{-iDt}e^{i\Gamma^{(1)}(0)} = U_0(t)U_\Gamma(t)U_D(t)U_\Gamma^\dagger(0). \tag{39}$$

Thus, by going into the interaction picture I have eliminated the complications that would arise due to time-ordering. This will make the rest of the process far easier than it would be otherwise.

4.1 Breaking up the evolution operator

To implement a block RG procedure, the evolution operator above must be broken into a product of an inter-block term (which describes the couplings between spins in adjacent blocks) and an intra-block term (which describes the couplings between spins within the same block). However unlike the procedure for the Floquet system where the form of the Hamiltonian makes this decomposition easy, performing the decomposition for the quasi-Floquet system is more complicated, due to the continuously varying Hamiltonian.

This is where the short-time and weak driving limits become critical. In order to be able to separate the evolution operator into an inter-block term and an intra-block term, I have to take the short-time and weak-driving limits. Due to the small size of t, the driving amplitudes, and $1/\boldsymbol{\omega}$, any commutators that would arise in the BCH formula when performing this partition can be effectively ignored to first order in t, the driving amplitudes, and $1/\boldsymbol{\omega}$. Thus, the evolution operator can be written as

$$U(t) \approx U_{0,inter}U_{0,intra}U_{\Gamma,intra}U_{D,intra}U^{\dagger}_{\Gamma_0,intra}U_{\Gamma,inter}U_{D,inter}U^{\dagger}_{\Gamma_0,inter}, \tag{40}$$

where the time dependence has been made implicit on the right-hand side, and $U_{\Gamma_0} = U_{\Gamma}(0)$. In the above $U_{0,intra}$ and $U_{0,inter}$ are

$$\exp\left[-it\sum_n (J_{2n-1}\sigma^z_{2n-1}\sigma^z_{2n} + h_{2n-1}\sigma^x_{2n-1})\right] \quad \text{and} \quad \exp\left[-it\sum_n (J_{2n}\sigma^z_{2n}\sigma^z_{2n+1} + h_{2n}\sigma^x_{2n})\right], \tag{41}$$

respectively, with similar expressions (i.e. expressions with the same index structure) for the other terms. These can now be grouped into an intra-block term and an inter-block term:

$$\mathcal{M} = U_{0,intra}U_{\Gamma,intra}U_{D,intra}U^{\dagger}_{\Gamma_0,intra} \quad \text{and} \quad \mathcal{N} = U_{\Gamma,inter}U_{D,inter}U^{\dagger}_{\Gamma_0,inter}U_{0,inter}. \tag{42}$$

These describe the dynamics within the blocks and among the blocks respectively. In the following section the intra-block term is used to define a new set of basis vectors, which are then used in the process of finding the renormalized parameters.

In the following section, I examine $\mathcal{M}$ with methods similar to those used for the Floquet system.

5 The intra-block term

In this section the intra-block term $\mathcal{M}$ is focused on. As mentioned before, this term is the evolution operator for the blocks. First, I will find its matrix elements, and then use the fact that this reduces to finding the eigenvalues and eigenvectors for a single spin pair $(\sigma_{2n-1}, \sigma_{2n})$ to find a more explicit of $\mathcal{M}$. This will then allow for the definition of a new set of basis vectors, which will be used to determine the renormalized parameters in section 6.

Upon expansion, the matrix elements of $\mathcal{M}$ can be represented as

$$\begin{aligned}\langle S'_1 ... S'_N|\mathcal{M}|S_1...S_N\rangle = \prod_{n=1}^{N/2} &\delta_{S'_{2n},S_{2n}}\langle S'_{2n-1}| \\ &\times \left[1 - it(J_{2n-1}\sigma^z_{2n-1}S_{2n} + h_{2n-1}\sigma^x_{2n-1})\right] \\ &\times \left[1 - i(A\sigma^z_{2n-1}S_{2n} + B\sigma^x_{2n-1} + C_{2n-1}\sigma^y_{2n-1}S_{2n})\right] \\ &\times \left[1 - it(D_{2n-1}\sigma^z_{2n-1}S_{2n} + E_{2n-1}\sigma^x_{2n-1} + F_{2n-1}\sigma^y_{2n-1}S_{2n})\right] \\ &\times \left[1 - i(A'\sigma^z_{2n-1}S_{2n} + B'\sigma^x_{2n-1} + C'_{2n-1}\sigma^y_{2n-1}S_{2n})\right] \\ &\quad\times |S_{2n-1}\rangle = \prod_{n=1}^{N/2}\delta_{S'_{2n},S_{2n}}\langle S'_{2n-1}|\mathcal{M}_{2n-1}(S_{2n})|S_{2n-1}\rangle,\end{aligned} \tag{43}$$

where A, B, and C_{2n-1} are the coefficients of the Pauli operators in (38), A', B', and C'_{2n-1} are the same coefficients with $t = 0$, D_{2n-1}, E_{2n-1}, and F_{2n-1} are the coefficients of the Pauli operators in (37), and $S_{2n} = \pm 1$ is the spin value for the even sites. From now on, I am going to suppress the indices on $C_{2n-1}, D_{2n-1}, \cdots$ to make the rest of the equations a little neater.

The problem has now reduced to finding the spectral decomposition of $\mathcal{M}_{2n-1}(S_{2n})$. Using the eigenvalues

$$\lambda^{\pm}_{2n-1} = \mathrm{Re}[\alpha] \pm i\sqrt{\mathrm{Im}^2[\alpha] + |\beta|^2} = \mathrm{Re}[\alpha] \pm i\Delta, \tag{44}$$

where α is

$$\begin{aligned}\alpha = &[(1 - itJ_{2n-1}S_{2n})(1 - iAS_{2n}) - ith_{2n-1}(CS_{2n} - iB)] \\ &\times [(1 - itDS_{2n})(1 - iA'S_{2n}) - (itES_{2n} + tFS_{2n})(C'S_{2n} - iB')] \\ &+ [(itJ_{2n-1}S_{2n} - 1)(CS_{2n} + iB) - ith_{2n-1}(1 + iAS_{2n})] \\ &\times [(tFS_{2n} - itES_{2n})(1 - iA'S_{2n}) + (1 + itDS_{2n})(C'S_{2n} - iB')]\,,\end{aligned} \tag{45}$$

and β is

$$\begin{aligned}\beta = &[(1 - itJ_{2n-1}S_{2n})(1 - iAS_{2n}) - ith_{2n-1}(CS_{2n} - iB)] \\ &\times [(itDS_{2n} - 1)(C'S_{2n} + iB') - (itES_{2n} + tFS_{2n})(1 + iA'S_{2n})] \\ &+ [(itJ_{2n-1}S_{2n} - 1)(CS_{2n} + iB) - ith_{2n-1}(1 + iAS_{2n})] \\ &\times [(itES_{2n} - tFS_{2n})(C' + iB') + (1 + itDS_{2n})(1 + iA'S_{2n})]\,,\end{aligned} \tag{46}$$

the spectral decomposition is

$$\mathcal{M}_{2n-1}(S_{2n}) = \lambda^{+}_{2n-1}|\lambda^{+}_{2n-1}(S_{2n})\rangle\langle\lambda^{+}_{2n-1}(S_{2n})| + \lambda^{-}_{2n-1}|\lambda^{-}_{2n-1}(S_{2n})\rangle\langle\lambda^{-}_{2n-1}(S_{2n})|. \tag{47}$$

To find the eigenvectors, I use the orthogonality condition and the spectral decomposition above. This gives the following eigenvectors:

$$|\lambda^{\pm}_{2n-1}(S_{2n})\rangle = \frac{|\beta|}{\sqrt{|\beta|^2 + (\Delta \mp \mathrm{Im}[\alpha])^2}}\left[|S_{2n-1} = +\rangle \pm \frac{i}{\beta}\left(\Delta \mp \mathrm{Im}[\alpha]\right)|S_{2n-1} = -\rangle\right]. \tag{48}$$

Now that I have these eigenvectors and eigenvalues, I can find an expression for $\mathcal{M}$. In the previous paragraph, I found the spectral decomposition of $\mathcal{M}_{2n-1}(S_{2n})$. However, this was relevant to only one spin: σ_{2n-1}. Since $\mathcal{M}$ describes all of the $(\sigma_{2n-1}, \sigma_{2n})$ pairs, I must find an expression that is relevant to a single pair of spins, and then take the product over all pairs. I will denote the matrix for a pair of spins by $\mathcal{M}_{2n-1}$. Note that the S_{2n} dependence has dropped out. The eigenvectors of this matrix are

$$|\lambda^{\pm}_{2n-1}(S_{2n})\rangle \otimes |S_{2n}\rangle = |\lambda^{\pm}_{2n-1}(S_{2n}), S_{2n}\rangle, \tag{49}$$

since $|\lambda^{\pm}_{2n-1}(S_{2n})\rangle$ are the vectors for the odd spin σ_{2n-1} and $|S_{2n}\rangle$ is the vector for the even spin σ_{2n}. These are the eigenvectors for a single block of spins in the σ^z_{2n} basis. With these vectors

$$\mathcal{M}_{2n-1} = \sum_{S_{2n}}\left(\lambda^{+}_{2n-1}|\lambda^{+}_{2n-1}(S_{2n}), S_{2n}\rangle\langle\lambda^{+}_{2n-1}(S_{2n}), S_{2n}| + \lambda^{-}_{2n-1}|\lambda^{-}_{2n-1}(S_{2n}), S_{2n}\rangle\langle\lambda^{-}_{2n-1}(S_{2n}), S_{2n}|\right), \tag{50}$$

and

$$\mathcal{M} = \prod_{n=1}^{N/2}\mathcal{M}_{2n-1}. \tag{51}$$

This operator describes the dynamics for all of the blocks of spins. The reason for its product form is that each block commutes with every other block, so if you think of each block as having its own 4-dimensional

Hilbert space then the space of the whole chain is just the tensor-product of all of these block spaces, with dimension 2^N. With this, a new basis can be defined in this subspace as

$$|M^{s_1\cdots s_{2n-1}\cdots}_{S_2\cdots S_{2n}\cdots}\rangle = \bigotimes_{n=1}^{N/2} |\lambda^{s_{2n-1}}_{2n-1}(S_{2n}), S_{2n}\rangle, \tag{52}$$

where $s_{2n-1} = \pm 1$ is a pseudo-spin that labels the eigenvectors. With this new basis, I will determine the matrix elements of $\mathcal{N}$ in the following section. And although the overall procedure will be similar to that for the Floquet system, it will be far more tedious, and the result will be far more complicated.

6 The inter-block term

In this section I focus on $\mathcal{N}$. This operator describes the dynamics of pairs of spins in adjacent blocks. Here, I examine the matrix elements of $\mathcal{N}$ in the aforementioned basis. This will allow us to determine the renormalized parameters, and the RG equations will follow. In this basis the matrix elements of $\mathcal{N}$ are

$$\begin{aligned}\langle M^{s_1\cdots s_{2n-1}\cdots}_{S_2\cdots S_{2n}\cdots}|\mathcal{N}|M^{s_1\cdots s_{2n-1}\cdots}_{S_2\cdots S_{2n}\cdots}\rangle &= \prod_{n=1}^{N/2}\langle\lambda^{s_{2n-1}}_{2n-1}(S_{2n}), S_{2n}|\left[1 - i(A'\sigma^z_{2n}\sigma^z_{2n+1} + B'\sigma^x_{2n} + C'\sigma^y_{2n}\sigma^z_{2n+1})\right] \\ &\times\left[1 - it(D\sigma^z_{2n}\sigma^z_{2n+1} + E\sigma^x_{2n} + F\sigma^y_{2n}\sigma^z_{2n+1})\right] \\ &\times\left[1 - i(A\sigma^z_{2n}\sigma^z_{2n+1} + B\sigma^x_{2n} + C\sigma^y_{2n}\sigma^z_{2n+1})\right] \\ &\times\left[1 - it(J_{2n}\sigma^z_{2n}\sigma^z_{2n+1} + h_{2n}\sigma^x_{2n})\right]|\lambda^{s_{2n-1}}_{2n-1}(S'_{2n}), S'_{2n}\rangle \\ &= \prod_{n=1}^{N/2}\mathcal{N}_n. \end{aligned}\tag{53}$$

(Remember, C, D, E, F, and C' have indices; here they are $2n+1$.) The next step is to calculate $\mathcal{N}_n$. After a tedious expansion, it is found to be

$$\mathcal{N}_n = \langle\lambda^{s_{2n-1}}_{2n-1}(S_{2n})|\left[S_{2n-2}\sigma^z_{2n-1}(\delta_{S_{2n},S'_{2n}}\Lambda_1 + \delta_{S_{2n},-S'_{2n}}\Lambda_2) + \delta_{S_{2n},S'_{2n}}\Omega_1 + \delta_{S_{2n},-S'_{2n}}\Omega_2\right]|\lambda^{s_{2n-1}}_{2n-1}(S'_{2n})\rangle, \tag{54}$$

where $\Lambda_{1,2}$ and $\Omega_{1,2}$ are long expressions that will become relevant later. The inner product $\langle\lambda^{s_{2n-1}}_{2n-1}(S_{2n})|\lambda^{s_{2n-1}}_{2n-1}(S'_{2n})\rangle$ and the matrix element $\langle\lambda^{s_{2n-1}}_{2n-1}(S_{2n})|\sigma^z_{2n-1}|\lambda^{s_{2n-1}}_{2n-1}(S'_{2n})\rangle$ must be calculated for $S_{2n} = S'_{2n}$, and for $S_{2n} = -S'_{2n}$. For these two cases the inner product is

$$\langle\lambda^{s_{2n-1}}_{2n-1}(S_{2n})|\lambda^{s_{2n-1}}_{2n-1}(S_{2n})\rangle = 1 \tag{55}$$

and

$$\begin{aligned}\langle\lambda^{s_{2n-1}}_{2n-1}(S_{2n})|\lambda^{s_{2n-1}}_{2n-1}(-S_{2n})\rangle &= \frac{|\beta||\beta'|}{\beta^*\beta'}\sqrt{\frac{[\beta^*\beta' + (\Delta - s_{2n-1}\mathrm{Im}[\alpha])(\Delta' - s_{2n-1}\mathrm{Im}[\alpha'])]^2}{[|\beta|^2 + (\Delta - s_{2n-1}\mathrm{Im}[\alpha])^2][|\beta'|^2 + (\Delta' - s_{2n-1}\mathrm{Im}[\alpha'])^2]}} \\ &= \frac{|\beta||\beta'|}{\beta^*\beta'}\sqrt{\frac{(\beta^*\beta' + G_nG'_n)^2}{(|\beta|^2 + G_n^2)(|\beta'|^2 + (G'_n)^2)}}\end{aligned}\tag{56}$$

respectively; here the primes indicate functions of $S'_{2n} = -S_{2n}$, $\Delta' = \sqrt{\mathrm{Im}^2[\alpha'] + |\beta'|^2}$, and $G_n = \Delta - s_{2n-1}\mathrm{Im}[\alpha]$. The matrix element is given by

$$\langle\lambda^{s_{2n-1}}_{2n-1}(S_{2n})|\sigma^z_{2n-1}|\lambda^{s_{2n-1}}_{2n-1}(S_{2n})\rangle = \frac{|\beta|^2 - (G_n)^2}{|\beta|^2 + (G_n)^2} \tag{57}$$

and

$$\langle\lambda_{2n-1}^{s_{2n-1}}(S_{2n})|\sigma_{2n-1}^z|\lambda_{2n-1}^{s_{2n-1}}(-S_{2n})\rangle = \frac{|\beta||\beta'|}{\beta^*\beta'}\sqrt{\frac{[\beta^*\beta' - (G_n)(G'_n)]^2}{[|\beta|^2+(G_n)^2][|\beta'|^2+(G'_n)^2]}}, \tag{58}$$

for each value of S'_{2n}. With these

$$\begin{aligned}\mathcal{N}_n = {} & S_{2n-2}(\delta_{S_{2n},S'_{2n}}\Lambda_1 + \delta_{S_{2n},-S'_{2n}}\Lambda_2)\left\{\frac{|\beta^2| - (G_n)^2}{|\beta^2| + (G_n)^2} + \frac{|\beta||\beta'|}{\beta^*\beta'}\sqrt{\frac{[\beta^*\beta' - (G_n)(G'_n)]^2}{[|\beta|^2+(G_n)^2][|\beta'|^2+(G'_n)^2]}}\right\} \\ & + (\delta_{S_{2n},S'_{2n}}\Omega_1 + \delta_{S_{2n},-S'_{2n}}\Omega_2)\left\{1 + \frac{|\beta||\beta'|}{\beta^*\beta'}\sqrt{\frac{[\beta^*\beta' + (G_n)(G'_n)]^2}{[|\beta|^2+(G_n)^2][|\beta'|^2+(G'_n)^2]}}\right\}.\end{aligned} \tag{59}$$

Due to this complicated form, finding the renormalized parameters will not be a straightforward task, like it was for the Floquet system. In fact, there are many possible renormalized parameters, and determining which ones to chose depends on some subtle details. The process of finding these terms will be shown in the following section.

7 Determining the renormalized parameters

Now that I have the matrix elements of $\mathcal{N}$, I can determine the renormalized parameters. These renormalized parameters will determine the RG equation, just like in the Floquet situation. This will then determine the possible phases for this 1-D quasi-Floquet Ising Model and its critical exponent. I will show, it is equal to the value found for the static and Floquet systems.

To determine the renormalized parameters, I associate with the matrix elements of $\mathcal{N}$ the matrix elements of a new matrix called $\mathcal{N}^R$, in the even-spin basis. $\mathcal{N}^R$ has the same form as $\mathcal{N}$, but it only involves the even spins σ_{2n} and it contains the renormalized couplings J^R_{2n-2} and fields h^R_{2n}. Its matrix elements are

$$\begin{aligned}\langle S_2,\cdots,S_{2n}|\mathcal{N}^R|S'_2,\cdots,S'_{2n}\rangle = \prod_{n=1}^{N/2}\Big\{&\delta_{S_{2n},S'_{2n}}\left[-itJ^R_{2n-2}(S_{2n-2}S_{2n}\tilde{\Lambda}_1 + \tilde{\Omega}_1) - ith^R_{2n}(S_{2n-2}S_{2n}\tilde{\Lambda}_2 + \tilde{\Omega}_2)\right] \\ &+ \delta_{S_{2n},-S'_{2n}}\left[-itJ^R_{2n-2}(S_{2n-2}S_{2n}\tilde{\Lambda}'_1 + \tilde{\Omega}'_1) - ith^R_{2n}(S_{2n-2}S_{2n}\tilde{\Lambda}'_2 + \tilde{\Omega}'_2)\right]\Big\},\end{aligned} \tag{60}$$

where $\tilde{\Lambda}_{1,2}$, $\tilde{\Omega}_{1,2}$, and their primes are very long terms that are not shown explicitly.

To determine J^R_{2n-2} and h^R_{2n}, I match terms in this equation with the matrix elements of $\mathcal{N}$. Even though $\mathcal{N}$ contains many terms, only a relatively small number of them can be considered in defining the renormalized parameters. The renormalized parameters must satisfy three conditions: 1. they must be real, 2. they must have one dimension of energy, 3. they must remain non-zero when the frequencies are taken to infinity. Here, I choose the following set of renormalized parameters:

$$J^R_{2n-2} = \frac{-tJ_{2n-2}J_{2n-1}s_{2n-1}}{\Delta - s_{2n-1}\mathrm{Im}[\alpha]} \quad \text{and} \quad h^R_{2n} = \frac{-t^2h_{2n}h_{2n-1}\mathcal{B}^2 f_0\tilde{f}_0}{\Delta - s_{2n-1}\mathrm{Im}[\alpha]}. \tag{61}$$

(Since I am working in units where $\hbar = 1$, t has dimensions of inverse energy.) It should also be noted that there are other possible sets of renormalized parameters that can be chosen at this order while still retaining the correct dimensions. However, they all ultimately lead to the same results for the critical exponent, and the criticality conditions that result from them all differ from that for the static system by a time-dependent logarithm. It should be noted that in the infinite-frequency limit, the results for the static system are not recovered exactly. This is not surprising, since several approximations were implemented. If an RG based on numerical methods was used, the results for the static system should be recovered, at least to a high degree of accuracy.

With the above renormalized parameters the RG equation is

$$\frac{|h_{2n}^R|}{|J_{2n-2}^R|} = \frac{t h_{2n} h_{2n-1} \mathcal{B}^2 f_0 \tilde{f}_0}{J_{2n-2} J_{2n-1}}. \tag{62}$$

This RG equation is proportional to the static system; therefore, I can use a similar technique to the one shown in [7]. The only difference is the proportionality factor: $t\mathcal{B}^2 f_0 \tilde{f}_0$. This factor adds an additive factor of $\ln t\mathcal{B}^2 f_0 \tilde{f}_0$ to μ' given by (25); however, this does not change the result for ν_{typ}. To obtain the correlation critical exponent, a technique similar to the one shown at the end of 3 is used. Here, $k_n = h_n/J_{n-1}$ and $k_n^R = h_n^R/J_{n-1}^R$. Despite the fact that the renormalized parameters are quite different from those for the static system, that the criticality condition resembles the one for the static system is one way that the results for the quasi-Floquet system in the high-frequency limit are similar to those for the static system.

In the following section, I discuss the results, and compare them to the results for the Floquet 1-D Ising Model.

8 Discussion

The main results of this RG procedure are the renormalized parameters, the RG equation, and the resulting critical exponent.

When one compares the renormalized parameters that I chose here to the those found for the 1-D Floquet Ising Model in [10], one sees that they are more complicated. This should not be surprising, considering the fact that the system studied here is more complicated and that several approximations were used here. Also, the RG equations quite different; however, they can be cast in the same form. The critical exponent that came from the RG equation for both the Floquet system and the quasi-Floquet system are identical. This means that the 1-D quasi-Floquet Ising Model is in the same universality class as its static and Floquet counterparts. One should also note that the RG equation that I derive here is proportional to that for the static system. This is not surprising, since in the high-frequency limit, the results for both of these systems should approach those for the static system. Another note is that unlike the Floquet situation [10], there is a large set of possible sets of renormalized parameters. However, only a few of them give non-trivial results when the frequencies are taken to all the way to infinity. In the Floquet situation, there is a small set of possible sets of renormalized parameters, and only one of them gives non-trivial results at this order.

9 Conclusion

In this study the 1-D quasi-Floquet Quantum Ising Model was considered in the short-time/high-frequency limit. This limit (when combined with the interaction picture and the Generalized Floquet-Magnus Expansion) made it reasonable to write the evolution operator in the form of a product of terms that resulted from the Generalized Floquet-Magnus Expansion, so that the RG procedure could be implemented. The resulting set renormalized parameters were selected from a large set of possible sets. It turns out that although the RG equations that result from the other sets differ from one another, the criticality conditions and critical exponents that follow from them are the all identical. The critical exponent that follows from these RG equations is $\nu = 1$; this is the same result found for the static and Floquet systems.

The RG procedure developed here is intended to be a starting point. The methods applied here are very restrictive in order to make the RG work analytically. Thus, numerical methods have to implemented if further progress is to be made. One possible way to use numerical methods could be used to determine the relevant quasi-energies, as was done for the Floquet Quantum Ising Model [1]. This could then lead to an analytical strong-disorder RG, which may point to a different pair of renormalized parameters. This will be the focus of a companion paper. It would also be interesting to see whether or not it is possible to develop a RG using an evolution operator that is derived numerically, which would allow more general frequencies longer time frames.

10 Acknowledgements

I am immensely thankful to my mentor James D. Whitfield, Casey Dowdle, Riley Chien, and Andrew Cupo for their support and feedback while I have worked on this project. I am also very thankful for the entire Whitfield Group for welcoming me into the group, giving me the opportunity to present this project to them at its various stages of completion, and allowing me to collaborate with them in several different arenas of physics and mathematics.

I would also like to thank my science research teachers Dr. Thomas Ticknor and Mr. Lawrence Sicheri for the work that they have done to help me.

References

[1] W. Berdanier, M. Kolodrubetz, S. Parameswaran, and R. Vasseur. Strong-disorder renormalization group for periodically driven systems. *Physical Review B*, 98(17):174203, 2018.

[2] A. N. Ciavarella, S. Caspar, H. Singh, M. J. Savage, and P. Lougovski. Floquet engineering heisenberg from ising using constant drive fields for quantum simulation. *arXiv preprint arXiv:2207.09438*, 2022.

[3] D. V. Else, W. W. Ho, and P. T. Dumitrescu. Long-lived interacting phases of matter protected by multiple time-translation symmetries in quasiperiodically driven systems. *Physical Review X*, 10(2):021032, 2020.

[4] A. J. Friedman, B. Ware, R. Vasseur, and A. C. Potter. Topological edge modes without symmetry in quasiperiodically driven spin chains. *Physical Review B*, 105(11):115117, 2022.

[5] V. Khemani, A. Lazarides, R. Moessner, and S. L. Sondhi. Phase structure of driven quantum systems. *Physical Review letters*, 116(25):250401, 2016.

[6] P. Liang, R. Fazio, and S. Chesi. Time crystals in the driven transverse field ising model under quasiperiodic modulation. *New Journal of Physics*, 22(12):125001, 2020.

[7] R. Miyazaki and H. Nishimori. Real-space renormalization-group approach to the random transverse-field ising model in finite dimensions. *Physical Review E*, 87(3):032154, 2013.

[8] R. Modak and T. Nag. Many-body localization in a long-range model: Real-space renormalization-group study. *Physical Review E*, 101(5):052108, 2020.

[9] C. Monthus. Many-body localization: construction of the emergent local conserved operators via block real-space renormalization. *Journal of Statistical Mechanics: Theory and Experiment*, 2016(3):033101, 2016.

[10] C. Monthus. Periodically driven random quantum spin chains: Real-space renormalization for floquet localized phases. *Journal of Statistical Mechanics: Theory and Experiment*, 2017(7):073301, 2017.

[11] A. Morningstar, D. A. Huse, and J. Z. Imbrie. Many-body localization near the critical point. *Physical Review B*, 102(12):125134, 2020.

[12] H. Nishimori and G. Ortiz. *Elements of phase transitions and critical phenomena*. Oup Oxford, 2010.

[13] M. Thakurathi, A. A. Patel, D. Sen, and A. Dutta. Floquet generation of majorana end modes and topological invariants. *Physical Review B*, 88(15):155133, 2013.

[14] S.-X. Zhang and H. Yao. Universal properties of many-body localization transitions in quasiperiodic systems. *Physical review letters*, 121(20):206601, 2018.